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Research statement

My research is in the subfield of **algebraic topology** known as **stable homotopy theory**, in connection with **algebraic K-theory**, **higher category theory** and **theoretical computer science**. I am particularly interested in showing how certain algebraic structures lead to new and effective computations. For instance, I have shown that model categories fail to accurately represent so-called **coalgebraic structures** in stable homotopy theory, and hence we need the language of ∞-categories to accurately capture homotopy coherent structures. More recently, I have studied **trace methods** to compute algebraic K-theory and introduced new variants of **topological Hochschild homology**. I have also extended my interests to theoretical computer science, applying methods of category theory and coalgebras to study **algebraic data type** interactions.

1 Motivation

How many solids exist where every face is the same regular polygon, and every corner looks the same? The answer to this question comes from ancient Greek mathematicians who showed that the only regular convex polyhedra are the five Platonic solids. The proof combines geometry with algebra, using the Euler characteristic χ, which relates the number of vertices V, edges E, and faces F via $\chi = V - E + F$. Euler proved that $\chi = 2$ for all convex polyhedra, and this condition, along with the symmetry of the Platonic solids, leads to the conclusion that only five such solids can exist.

The Euler characteristic's success in classifying these solids was an early example of algebraic invariants solving geometric problems, leading to its generalization in algebraic topology. Topologists extended χ to higher-dimensional spaces, or *CW-complexes*, which are constructed similarly to polyhedra, but with n-dimensional cells. In homotopy theory, spaces are studied up to weak homotopy equivalence, reducing them to CW-complexes. This lead to the motivating, and notoriously difficult, problem of computing homotopy groups. Even for simple spaces like spheres, these groups remain unknown but contain profound information, such as connections to the Riemann zeta function [Ada66], or the classification of exotic smooth structures on spheres [KM63].

To aid in these computations, algebraic topologists developed more generalized invariants, such as cohomology and topological K-theory, all unified by Brown's representability theorem. A *spectrum* is a sequence of topological spaces connected by loop space equivalences, used to represent generalized homology and cohomology theories. For instance, cohomology groups $H^n(X;\mathbb{Z})$ are represented by the Eilenberg-Mac Lane spectrum, which encodes algebraic information about the space. This idea is the starting place of stable homotopy theory, in which we study spectra instead of spaces. The sphere spectrum $\mathbb{S}$ represents stable homotopy groups π_*^S, which reveal geometric features across dimensions, and understanding these groups classify smooth manifolds up to framed cobordism, as shown by the Thom–Pontryagin theorem [Tho54].

Spectra also play a foundational role in algebraic K-theory, first defined by Grothendieck and later expanded by Quillen and Waldhausen. This theory generalizes the Euler characteristic, encoding a universal additive invariant. Algebraic K-theory is rich with complexity and encodes deep geometric and arithmetic information, from topological properties of CW-complexes [Wal65] to the prime factorization of class numbers for cyclotomic fields [Kur92]. Although algebraic K-theory is challenging to compute, connections to other invariants, such as Hochschild homology (HH) and its refinements, such as topological Hochschild homology (THH), offer effective approximations.

2 Overview of my research

My research focuses on algebraic structures in stable homotopy theory using the modern language of ∞-categories. These structures provide more tools for computations and I am also interested in applying this framework to obtain new approximations of algebraic K-theory. For instance, many rings R have the additional structure of a Hopf algebra, which means there is a comultiplication map $R \to R \otimes R$ that records all possible ways an element in R can be assembled. More generally, an abelian group with a comultiplication is referred as a *coalgebra* and these objects appear throughout topology and geometry. For example, the singular chain complex $C_*(X;\mathbb{Z})$ of a space X is a coalgebra that encodes the cellular homology $H_*(X;\mathbb{Z})$. It records how all the n-dimensional cells of the space X have been obtained as the combination of p and q-cells where $p+q=n$.

In joint project with Shipley [PS19], we show that the usual point-set model methods used to describe ring structures in spectra fail to capture coalgebras in spectra, and I later demonstrate how ∞-categories circumvent this issue [Pér22b]. This provide a framework to implement these new algebraic structures, resulting to more refined invariants for spaces. I also show how the homotopy theory of ring spectra is encoded in coalgebras [Pér22a], allowing to gauge how far a homomorphism of spectra preserve multiplications, see Section 3.1.1. In joint work with Beardsley, we prove a Koszul duality that relates comodules and modules in higher topoi: formal generalizations of spaces [BP23b]. Our approach generalizes the recognition of loop spaces as certain grouplike spaces with concentrated homotopy groups between certain values. I show in [Pér24b, Pér24a] that certain model categories can still accurately capture the homotopy theory of comodules, providing a new coalgebraic description of Waldhausen's rational A-theory of spaces, this is discussed in Section 3.1.2.

In my work discussed in Section 3.2, I describe how to vary classical trace methods to gain new insight into algebraic K-theory. I show that coalgebras in spectra also have a coalgebraic Dennis trace on topological coHochschild homology (coTHH), an analogous invariant of coalgebras introduced in [HS21], see Section 3.2.1. With Bayındır, we show how THH and coTHH relate to one another [BP23a] and we compute the coTHH of the Steenrod algebra spectrum. In joint work with Klanderman [KP23], we provide two coalgebraic extensions of the Hattori–Stallings trace, and introduce new algebraic K-theories for coalgebras. In joint work with Gerhardt and Soré, we relate the usual algebraic K-theory with their coalgebraic analogues [GPS]. These new theories provide a new framework to compute certain classical algebraic K-theories of rings such as the power series $k[[x]]$ or the group ring kG of a group G. In on-going joint work with Brazelton, Calle, Chan and Keenan, we aim to extend those trace methods in stable homotopy theory in our characterizations of Thom spectra as comodules. We expect that algebraic K-theory will help encode how far certain comodules are to be Thom spectra. This project is funded by the SQuaRE program of the American Institute of Mathematics. With Angelini-Knoll and Merling, we construct a quaternionic refinement of THH [AMP24] which lead to new trace methods with connection to Floer homology, see Section 3.2.2. In on-going joint work with Keenan, we generalize the multiplicative compatibility of the Dold–Kan correspondence to the stable homotopy theory setting and analyze filtered refinements of THH, see Section 3.2.3. These lead to a formalization of appearances of Leibniz rule in spectral sequences which are extremely useful for computations of many algebraic invariants.

In joint work with North [NP23], described in Section 3.3, we demonstrate how to encode universally partial induction arguments. Specifically, we use the initial algebra of coalgebraic enrichment applied to algebraic data type interactions. In [MNP24], we extend our framework to describe a multitude of new partial types, for instance, of booleans, of lists, of strings, of trees, etc.

3 Past, current, and future projects

3.1 Coalgebraic structures in stable homotopy theory

Stable homotopy theory extends classical algebra by replacing abelian groups with spectra. A spectrum with richer algebraic structure allows us to more deeply understand its homotopy groups. For example, if R is a ring spectrum, then $\pi_*^S(R)$ is a graded ring. However, verifying additional algebraic structures on spectra is challenging due to complex homotopy coherences. While model categories provided a point-set framework for modules and rings in spectra [EKMM97, HSS00, MMSS01, MM02], Lewis showed that there will always be limited [Lew91]. Despite the success of the Goerss–Hopkins–Miller theorem, which uses coalgebraic methods to link elliptic curves and ring spectra [GH04], model categories of spectra struggle to handle coalgebras adequately.

Objective. Define comultiplications $C \to C \otimes_{\mathbb{S}} C$ in spectra and extend methods from modules and rings to comodules and coalgebras.

3.1.1 Coalgebras in spectra

In joint work with Brooke Shipley, we show that model categories of spectra poorly support coalgebras beyond trivial cases like those arising from spaces. Specifically, the diagonal map on a space X defines a comultiplication on the stabilization $\mathbb{S}[X] \to \mathbb{S}[X \times X] \simeq \mathbb{S}[X] \otimes_{\mathbb{S}} \mathbb{S}[X]$, but:

Theorem 1 ([PS19])**.** *Model categories of spectra can only define coalgebras essentially of the form* $\mathbb{S}[X]$.

This is problematic since non-cocommutative examples exist, such as the stable linear dual of a finite, non-abelian topological group, $\mathbb{S}[G]^\vee$. This coalgebra structure cannot be captured by model categories. Therefore, prior methods for rings and modules cannot be directly adapted for coalgebras. The work of Joyal and Lurie on ∞-categories [Joy08, Lur17] overcomes this limitation. By reversing the arrows in ∞-categories, we define a homotopy theory of coalgebras in spectra, leading to the following result:

Theorem 2 ([Pér22b, 5.5, 5.6])**.** *Coalgebras and cocommutative coalgebras in spectra cannot be described within usual model categories of spectra.*

This behavior extends to derived settings, showing that coalgebras require the language of ∞-categories for their study [Pér22b, 4.3].

Understanding the homotopy theory of coalgebras is key to advancing our knowledge of rings and modules. May's recognition theorem [May72] shows that certain groups in the homotopy category of spaces are equivalent to loop spaces ΩY, while abelian groups correspond to ∞-loop spaces, an instance of Koszul duality. The proof is based on the natural coalgebraic structure given by the diagonal $X \to X \times X$ on each space X. In [BP23b], Bearsdley and I extend the proof to spaces that have homotopy groups concentrated to certain levels. I have shown a novel connection between coalgebras and ring spectra:

Theorem 3 ([Pér22a])**.** *The homotopy theory of ring spectra is enriched in the homotopy theory of coalgebras in spectra, capturing all partial ring homomorphisms.*

Details on partial ring homomorphisms are in Section 3.3. The above result in particular recovers the tangent complex of a ring, crucial in obstruction theory, as the primitive elements of a coalgebra spectrum.

3.1.2 Comodules in spectra

The homotopy theory of HR-modules in spectra is equivalent to the derived category $\mathcal{D}(R)$ of R-modules, with $\pi_*^S(M \otimes_{HR} N) \cong \mathrm{Tor}_*^R(M, N)$. However, comodules in spectra present challenges for defining a relative cotensor product, a key tool in Hopf–Galois theory. I addressed these issues for specific cases:

Theorem 4 ([Pér24b])**.** *Let R be a commutative ring with global dimension zero, and C a simply connected dg coalgebra over R. The homotopy theory of C-comodules in connective HR-spectra is equivalent to the derived category of C-comodules in non-negative chain complexes.*

I have also defined a derived cotensor product for C-comodules, $M \square_C N$, with $\pi_*^S(M \square_C N) \cong \mathrm{coTor}_*^C(M, N)$, which provides a multiplicative structure when C is cocommutative. These results apply directly to algebraic K-theory, or A-theory, of spaces. For a connected space X, Waldhausen's A-theory of X relates to $K(\mathbb{S}[\Omega X])$ [Wal85] and pseudo-isotopy theory [WJR13]. Combining my previous results, I describe rational A-theory:

Theorem 5 ([Pér24a])**.** *Let $C_*(X; \mathbb{Q})$ be the rational singular chain complex of a simply connected space X. The rational A-theory of X is equivalent to the K-theory of perfect non-negative chain complexes with a $C_*(X; \mathbb{Q})$-coaction: $A(X, H\mathbb{Q}_*) \simeq K(\mathrm{coMod}^{\mathrm{perf}}_{C_*(X,\mathbb{Q})})$.*

This new description enables a coalgebraic version of the Dennis trace, which is the another focus of my research.

3.2 Trace Methods in Algebraic *K*-theory

Computing the algebraic K-theory of a ring R is notoriously difficult. A more tractable invariant is Hochschild homology $\mathrm{HH}_*(R)$, which can be defined as $\mathrm{Tor}_*^{R \otimes R}(R, R)$. Hattori and Stallings (1965) showed that any trace map on matrices over R factors through $\mathrm{HH}_0(R)$, recording a rank assignment via the Hattori–Stallings trace $K_0(R) \to \mathrm{HH}_0(R)$. For ring spectra R, Hochschild homology is replaced by topological Hochschild homology $\mathrm{THH}(R)$, introduced by Bökstedt [CLM+20]. This spectrum captures traces and cyclic invariants, formalized using model categories [EKMM97] and ∞-categories [NS18]. The universal additivity of algebraic K-theory yields the *Dennis trace*, a map $K(R) \to \mathrm{THH}(R)$, extending the Hattori–Stallings trace and providing a useful approximation for computing K-theory.

3.2.1 A Coalgebraic Dennis Trace

Topological coHochschild homology (coTHH), introduced by Hess–Shipley [HS21], is an analogue of THH for coalgebras in spectra, extending coHochschild homology (coHH) computed as a co-Tor [Doi81]. They show an equivalence $\mathrm{THH}(\mathbb{S}[\Omega X]) \simeq \mathrm{coTHH}(\mathbb{S}[X])$ under certain connectivity assumptions on X, leading to a trace $A(X) \to \mathrm{coTHH}(\mathbb{S}[X])$.

Furthermore, they prove $\mathrm{coTHH}(\mathbb{S}[X]) \simeq \mathbb{S}[\mathcal{L}X]$, where $\mathcal{L}X$ is the free loop space of X. The study of $\mathcal{L}X$ is significant in topology and physics, as demonstrated in the works of Gromell–Meyer [GM69] and in string topology [CJ02]. Recent results by Bohmann–Gerhardt–Shipley [BGS21] further explore the homology of free loop spaces, using a coBökstedt spectral sequence [BGH+18]. However, Hess–Shipley's work is restricted to coalgebras of the form $\mathbb{S}[X]$ due to limitations in the model category framework.

Objective. Extend coTHH to the entire homotopy theory of coalgebras in spectra, applying K-theoretic methods to coalgebras. This would enable new computations of $\mathrm{coTHH}(C)$ for coalgebras beyond $\mathbb{S}[X]$.

Coalgebraic Trace in the Dualizable Setting (joint with Haldun Özgür Bayındır) As a first step to achieving the goal above, we define the coTHH of any coalgebra in spectra. Our approach builds on [HS21] and is dual to the method in Nikolaus–Scholze [NS18], allowing us to explore new coalgebras in spectra. For a spectrum E, let $E^\vee = \hom(E, \mathbb{S})$ be its stable linear dual. Under specific finiteness conditions, this duality connects the homotopy theories of coalgebra spectra and ring spectra.

Theorem 6 ([BP23a, 1.1, 4.4])**.** *For any coalgebra spectrum C, assuming certain finiteness, there is an equivalence* $\mathrm{coTHH}(C) \simeq \mathrm{THH}(C^\vee)^\vee$*, and the Dennis trace becomes* $K(C^\vee) \to \mathrm{coTHH}(C)^\vee$.

Our main example of computations is the coTHH of the Steenrod algebra spectrum encoding all operations in cohomology [BP23a, 1.3]. This is the first coTHH computation for a coalgebra beyond the form $\mathbb{S}[X]$, with additional examples provided in our work.

K-theories for Coalgebras The Dennis–Hess–Shipley trace $A(X) \to \mathrm{coTHH}(\mathbb{S}[X])$ and the trace $K(C^\vee) \to \mathrm{coTHH}(C)^\vee$ from Theorem 6 are initial steps towards a coalgebraic Dennis trace but still rely on connections to rings. In this project, we develop traces purely within the coalgebraic setting, extending the Hattori–Stallings trace $K_0(R) \to \mathrm{HH}_0(R)$. Quillen [Qui88] showed that for a coalgebra C, $\mathrm{coHH}_0(C)$ is the universal home for cotrace maps, the coalgebraic analogue of traces for rings. We define $K_0^c(C)$ as a coalgebraic K_0-group and extend cotrace maps $V \to \mathcal{M}_n(C)$ to $K_0^c(C)$.

Theorem 7 ([KP23, 1.2])**.** *For a commutative ring R of global dimension zero and an R-coalgebra C, there is a Hattori–Stallings cotrace* $K_0^c(C) \to \mathrm{coHH}_0(C)^*$.

This map mirrors the Dennis trace, allowing comparison between coalgebraic K-theory $K^c(C)$ and the algebraic K-theory of C^*, the dual algebra.

Theorem 8 ([GPS])**.** *For certain coalgebras C, there is a comparison map $K^c(C) \to K(C^*)$ that becomes an equivalence for certain coalgebras, and is compatible with the (co)traces.*

For instance, the map induces an equivalence $K^c(\mathbb{F}[x]) \simeq K(\mathbb{F}[[x]])$ for a field $\mathbb{F}$ of characteristic zero. Similarly, we define a coalgebraic G-theory $G^c(C)$ for finite-dimensional comodules, analogous to Swan's G-theory of finite-dimensional representations. Recall that given any algebra A, there is a homomorphism $\chi\colon G_0(A) \to A^*$ defined by the characters of a group representation

Theorem 9 ([KP23, 1.4], [GPS])**.** *There is a character map $\chi^c\colon G_0^c(C) \to \mathrm{coHH}_0(C)$, and a comparison map $G^c(C) \to G(C^*)$ that becomes an equivalence for certain coalgebras, compatible wth the characters.*

In future work with Sanjana Agarwal and David Mehrle, we aim to define higher traces $K_n^c(C) \to \mathrm{coHH}_n(C)^*$ and $G_n^c(C) \to \mathrm{coHH}_n(C)$ for all $n \geq 0$.

Coalgebraic characterization of Thom spectra (joint work with Thomas Brazelton, Maxine Calle, David Chan and Liam Keenan) This project is funded by the SQuaRE program of the American Institute of Mathematics. In 1950, Pontryagin showed that the n-th framed cobordism group of smooth manifolds is precisely $\pi_n^S(\mathbb{S})$. In 1954, Thom extended this result, showing that all variations of cobordism are represented by what are now called *Thom spectra* [Tho54]. For example, the singular homology spectrum $H\mathbb{F}_2$ is a Thom spectrum [Mah79], while connective topological K-theory is not [Mah87, AHL09].

How can we identify spectra that arise as a Thom spectrum? A Thom spectrum Mf is specified by a universal bundle over a manifold X, and Mf is a comodule spectrum over $\mathbb{S}[X]$ via the Thom diagonal. The coaction $Mf \to Mf \otimes_{\mathbb{S}} \mathbb{S}[X]$ is an essential feature of the Thom spectrum, and a necessary component in the celebrated Thom isomorphism, see [ABG$^+$14, Bea23].

Objective. Give an algebraic criterion for an $\mathbb{S}[X]$-comodule spectrum to be equivalent to a Thom spectrum. Extend the cotrace of Theorem 7 into a map $K^c(\mathbb{S}[X]) \to \mathrm{coTHH}(\mathbb{S}[X])^\vee$ and obtain its relative analogue to any Thom spectrum $\mathrm{coTHH}(\mathbb{S}[X], Mf) \to K^\vee(\mathbb{S}[X], Mf)$. Extend Dennis–Hess–Shipley trace $A(X) \to \mathrm{coTHH}(\mathbb{S}[X])$ with no connectivity assumption on X.

Our methods extend the approach of Hess–Shipley [HS16] to the ∞-categorical setting, and we show that the homotopy theory of $\mathbb{S}[X]$-comodule spectra embeds in the homotopy theory of parametrized spectra over X. This perspective is very useful as we can show that a Thom spectrum Mf must be in Pic of $\mathbb{S}[X]$-comodules: invertible comodules with respect to a multiplicative structure. We also expect to further extend the duality of Hess-Shipley $\mathrm{THH}(\mathbb{S}[\Omega X]) \simeq \mathrm{coTHH}(\mathbb{S}[X])$ into a relative variation that would consider Thom spectra comodules.

3.2.2 A Quaternionic Refinement of the Dennis Trace (joint with Gabriel Angelini-Knoll and Mona Merling)

If R is a ring spectrum, $\mathrm{THH}(R)$ has an S^1-action that refines the Dennis trace by considering cyclic analogues of THH, which better approximate algebraic K-theory. This S^1-action arises from defining THH as a cyclic spectrum, the universal home of trace maps [CLM$^+$20]. Fiedorwicz–Loday [FL91] introduced *crossed simplicial groups*, usually denoted $\Delta\mathbf{G}$, defined from a collection of groups $\mathbf{G}_\bullet$ which associate a topological group$|\mathbf{G}_\bullet|$, and give rise to new homological invariants with $|\mathbf{G}_\bullet|$-actions. For the cyclic groups $\mathbf{C}_\bullet$, this recovers classical Hochschild homology, but for other CSGs, these invariants define more refined homologies.

Theorem 10 ([AMP24])**.** *For a CSG $\Delta\mathbf{G}$ and a ring spectrum R with twisted G_0-action, there exists an associated topological homology theory $\mathrm{TH}\mathbf{G}(R)$ with a $|\mathbf{G}_\bullet|$-action.*

We introduce a quaternionic refinement called *quaternionic topological Hochschild homology* ($\mathrm{TH}\mathbf{Q}(R)$). The table below summarizes the main examples, including the usual topological Hochschild homology and real topological Hochschild homology (THR).

CSG	$\Delta\mathbf{G}$	G_n	$\lvert\mathbf{G}_\bullet\rvert$	$\mathrm{TH}\mathbf{G}$
cyclic	$\Delta\mathbf{C}$	C_{n+1}	S^1	THH
dihedral	$\Delta\mathbf{D}$	$D_{2(n+1)}$	$O(2)$	THR
quaternionic	$\Delta\mathbf{Q}$	$Q_{4(n+1)}$	$\mathrm{Pin}(2)$	$\mathrm{TH}\mathbf{Q}$

For a connected space X with a C_4-action, we prove a $\mathrm{Pin}(2)$-equivariant equivalence between $\mathrm{TH}\mathbf{Q}(\mathbb{S}[\Omega X])$ and $\mathbb{S}[\mathcal{L}^\tau X]$. Here, a C_4-action $t\colon X \to X$ induces a C_2-action $\tau\colon X \to X$ where

$\tau^2 = 1$, and $\mathcal{L}^\tau X$ is the twisted free loop space formed by loops $\gamma\colon [0,1] \to X$ such that $\gamma(1) = \tau(\gamma(0))$. One can check that we can identify $\mathcal{L}^\tau X$ with the space of left C_4-equivariant maps $\mathrm{Pin}(2) \to X$. This Pin(2)-equivariance is also central to Manolescu's work on equivariant Floer homology [Man16]. In future work, we will define higher Pin(2)-cyclotomic structures for $\mathrm{TH}\mathbf{Q}(R)$, analogous to the $O(2)$-cyclotomic structure of $\mathrm{THR}(R)$ [Dot12, Hø16, HM15]. This will lead to a quaternionic refinement of the Dennis trace. We will also adapt Merling's equivariant algebraic K-theory [Mer17] to incorporate this twisting action, further enriching the theory.

While certain topological groups in our examples are contractible, these variations still yield rich new invariants. In future projects, we will compute equivalences for *topological symmetric homology* and *hyperoctahedral homology*, providing new insights for finite groups with anti-involution.

3.2.3 Dold-Kan Correspondence and Filtered Variations on Topological Hochschild Homology (joint work with Liam Keenan)

Given an abelian group A, a filtration F_* consists of a sequence of injective homomorphisms $F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow A$, with associative and unital maps $F_n \otimes_{\mathbb{Z}} F_m \to F_{n+m}$ inducing a ring structure on A. A cofiltration F^*, on the other hand, involves surjective homomorphisms, and the structure mirrors that of the filtration. The Dold–Kan correspondence shows that simplicial abelian groups are equivalent to connective chain complexes. Although these categories have different multiplicative structures, in homotopy theory, these structures are equivalent, a fact established by Eilenberg and Mac Lane and formalized in model categories [SS03].

Lurie [Lur17] extends the Dold-Kan correspondence between (co)simplicial spectra and (co)filtered spectra via the skeleton filtration sk_* for simplicial spectra, and similarly defines coskeleton cofiltrations cosk^* for cosimplicial spectra. However, his work does not directly address how these constructions interact with multiplicative structures.

Theorem 11. *The skeleton filtration preserves multiplicative structures. For a simplicial ring spectrum R, there is an induced multiplication $\mathsf{sk}_n(R) \otimes \mathsf{sk}_m(R) \to \mathsf{sk}_{n+m}(R)$, derived from a higher categorical Eilenberg–Zilber map.*

We aim to extend this framework to coalgebraic structures and coskeleton filtrations. Our motivation is to study the multiplicative structures on spectral sequences associated with these filtrations. For instance, the E^1-page of the R-based Adams spectral sequence has only an associative structure but plays a key role in computing stable homotopy groups of spheres. In particular, we investigate May's filtration of THH [AKS18]. We expect to show that the filtered THH of the coskeleton cofiltration of the Amistur cosimplicial complex of a commutative ring homomorphism $A \to B$ is the $\mathrm{THH}(B)$-nilpotent completion of $\mathrm{THH}(A)$.

3.3 Measuring Partial Algebraic Interactions (joint with Paige Randall North)

Coalgebras provide a surprising framework for studying partial algebraic structures. For example, given a coalgebra C on a set C, a function $f_c\colon M \to N$ between sets M and N is called a *partial homomorphism* if $f_c(xy) = f_{c_1}(x) f_{c_2}(y)$ for some decomposition $c \mapsto (c_1, c_2)$. This leads to a coalgebra $\{M, N\}$ of partial homomorphisms, extending the classical notion of homomorphisms to a partial setting [Swe69, HLV17]. In stable homotopy theory, I have generalized this to enrich algebras in spectra [Pér22a].

This idea bridges algebraic and coalgebraic structures, which have deep connections in theoretical computer science. Inductive data types correspond to algebras, while coinductive data types are governed by coalgebras. For instance, any Turing machine can be modeled as an automaton, and coinductive methods capture the possible transitions and accepted states of the machine. This approach is foundational in projects like CALF and Concurrent Kleene Algebra (CoNeCo).

Our work connects these ideas by exploring the algebraic interaction of successors and predecessors. Successor algebras encode natural numbers via maps $A+1 \to A$, while predecessor coalgebras use maps $C \to C+1$ to track how far a process is from completion. We show that a collection of partial homomorphisms between such structures measures how closely the homomorphisms approach totality, and we encode this as a coalgebra $\{A, B\}$.

Theorem 12 ([NP23, Example 15, Theorem 31])**.** *Let $\mathbb{n} = \{0, 1, \ldots, n\}$ be the subalgebra of $\mathbb{N}$, for which the successor of n is itself. Then $\mathbb{n}$ is the initial algebra of all partial homomorphisms that are successful after n-successions.*

This result highlights how partial induction can be modeled algebraically and coinductively, offering a new perspective on processes that gradually approach completion. In [MNP24], we extend this framework to a large family of algebraic data types known as W-types.

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